Three Problems with Dynamic Hedging in Discrete Time

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I. Introduction

While accepting the entire Black Scholes framework, we set up the equations for dynamic hedging in "real life", that is how the limiting process works, and show that in real time Δt there is an added variance term from the randomness of the exposure over the life of an option, which we extract. The key problem is that we are not conditioning on specific hedge ratios.

- First Problem, while theta= gamma on balance, the gamma level is stochastic, since a given option will not be at the same strike all the time, which makes the variance of the package much higher than usually calculated.
- Second problem: the same applies to the stream of delta hedges, since delta is stochastic, though uniformly distributed.
- Severe Problem, am information-theoretic one, which makes the two previous ones relevant: transactions have an additional lag, causing a second effect in discretization of stochastic integral.

II. The operation of dynamic hedging

We reexpress the dynamic hedging argument, which requires the ability to reduce the variance of the portfolio through continuous revision of the holdings in the underlying asset.

Consider the Black and Scholes package, as a discretization of the stochastic integral: the payoff of a call option corresponds to initial price at which it was sold, C_0, the expectation of terminal "moniness" $\mathbb{E}\left((S_t-K)^+\right)$, plus the performance of $\Psi(\Delta t)$ the stream of dynamic hedges between t_0 and t (which includes the initial "delta" hedge). We break up the period (t_0,t) into n increments of size Δt. Here, the hedge ratio $\frac{\partial C}{\partial S}$ is computed as of time $t_0+(i-1)\Delta t$, but we get the nonanticipating difference between the price at the time the hedge was initiated and the resulting price at $t_0+i\,\Delta t$.

Hence the first equation:

$$\overline{\Pi}_n = C_0-(S_t-K)^+ + \Psi(\Delta t) \tag{1}$$

where $n=\frac{t-t_0}{\Delta t}$ and

$$\Psi(\Delta t)=\sum_{i=1}^{n}\frac{\partial C}{\partial S}|_{S_{t_0+(i-1)\Delta \mathrm{t}},t_0+(i-1)\Delta \mathrm{t}}\big(S_{t_0+i\Delta \mathrm{t}}-S_{(t_0+(i-1)\Delta t)}\big)$$

This operation is supposed to make the payoff deterministic at the limit of $\Delta t\to 0$, with a net profit or loss of $\Psi(\Delta t)-(S_t-K)^+$ that converges to equality with C_0. This net difference corresponds to an Ito-McKean stochastic integral, under a set of assumptions about the dynamics of the process. More formally, under the right dynamics, by the law of large numbers, $\overline{\Pi}_n$ converges:

$$\overline{\Pi}_n \ \xrightarrow{P}\ 0 \qquad \text{when } n\to\infty.$$

That is to say that for any positive number ϵ,

$$\lim_{n\to\infty}\mathbb{P}\big(\,|\overline{\Pi}(\Delta t)+C_0-(S_t-K)^+|>\varepsilon\,\big)=0.$$

Over a single period Δt, the conditional mistracking in changes in the delta-hedged portfolio are as follows:
The "gamma",

$$\Gamma_{t_0+i\,\Delta t}=-\frac{1}{2}\frac{\partial^2 C}{\partial S^2}|_{S_{t_0+(i-1)\Delta \mathrm{t}},t_0+(i-1)\Delta \mathrm{t}}\big(S_{t_0+i\Delta \mathrm{t}}-S_{(t_0+(i-1)\Delta t)}\big)^2 \tag{2}$$

which is offset by the "theta"

$$\theta_{t_0+i\,\Delta t}=\frac{\partial C}{\partial t}|_{S_{t_0+(i-1)\Delta \mathrm{t}},t_0+(i-1)\Delta \mathrm{t}}\Delta t \tag{3}$$

and the performance at a time period is $\theta_{t_0+i\,\Delta t}-\Gamma_{t_0+i\,\Delta t}$.

We are burdening equations 1 and 2 with notational precisions between periods to show the non-overlapping of periods between the "gamma" and the subsequent performance of the underlying asset.

Most analyses assume the gamma (second derivative) given, so the dynamic hedge is conditional to a given strike, S_t, and time to expiration. But what if we were, in addition, to take a realization among *all* dynamic hedges?

III. The stochastic exposure problem, case of gamma

So here we have two random terms, the change in S and the distribution of the theta/gamma, both independent since they address —by the rules of stochastic calculus—sequential, non-overlapping periods, as made clear in the equations. Now consider that the gamma of the option is the scaled density of a Gaussian variable, hence:

We have ζ the new random variable, $\zeta=\theta_{t_0+i\,\Delta t}-\Gamma_{t_0+i\,\Delta t}$:

$$\zeta=\frac{z\,S_t\,\sigma\,\Delta t}{2\sqrt{t-t_0}}-\frac{z\,(S_t-S_{t+\Delta t})^2}{2\,S_t\sigma\,\sqrt{t-t_0}}$$

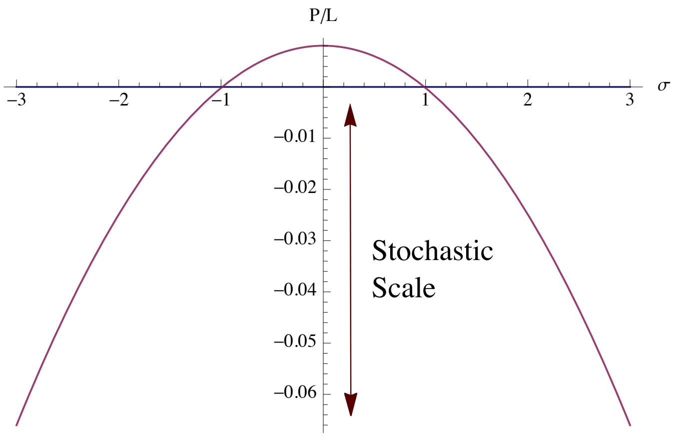

Figure 1. The P/L from hedges is stochastic, so is exposure in a stream of hedges, so we multiply the scaled P/L (variable z_1) by another variable corresponding to the scaling, z, for a product ζ.

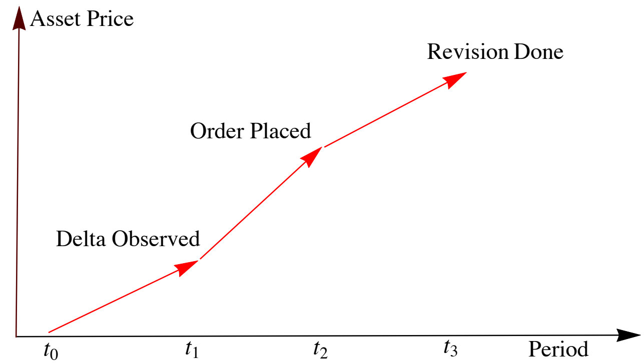

Figure 2. **The Impossibility of Dynamic Hedging.** *Additional informational lag in dynamic hedging prevents the smoothing of ΔS^2 at the limit of Δt going to 0, adding another term to the Ito-McKean Integral.*

To separate the random variables:

$$\zeta = \frac{z\left(S_t^2 \sigma^2 \Delta t - (S_t - S_{t+\Delta t})^2\right)}{2\, S_t\, \sigma \sqrt{t - t_0}}$$

Condensed:

$$\zeta = \frac{z(z_1 - 1)}{c}$$

where

$$z_1 = \left(\frac{S_t - S_{\Delta t + t}}{S_t \sigma \sqrt{\Delta t}}\right)^2$$

For z, it can be shown that its probability distribution and support are:

$$p(z) = \frac{2}{\sqrt{-2\log(z) - \log(2\pi)}},\ z \in \left(0, \frac{1}{\sqrt{2\pi}}\right).$$

(and 0 elsewhere).

Note that

$$\left(\frac{S_t - S_{\Delta t + t}}{S_t \sigma \sqrt{\Delta t}}\right) \sim \text{Gaussian}(0,1)$$

and the nonstochastic part c:

$$c = -\frac{2\sqrt{t - t_0}}{S\, \sigma\, \Delta t}$$

$$z_2 = \frac{z_1 - 1}{c} \sim \text{scaled Chi-square}$$

The density for z_1 ,$p(z_1) = \frac{e^{-\frac{z_1}{2}}}{\sqrt{2\pi}\sqrt{z_1}}$, $z_1 \in [0, \infty)$. which makes the density for z_2

$$p(z_2) = \frac{c\, e^{-\frac{c\, z_2}{2} - \frac{1}{2}}}{\sqrt{2\pi}\sqrt{c z_2 + 1}}\ ,\ z_2 \in [-\frac{1}{c}, \infty). \tag{4}$$

We end up with:

$$p(\zeta) = \begin{cases} \int_0^{\frac{1}{\sqrt{2\pi}}} \frac{\sqrt{\frac{2}{\pi}}|c| e^{-\frac{c\zeta}{2z} - \frac{1}{2}}}{z\sqrt{-2\log(z) - \log(2\pi)}\sqrt{\frac{c\zeta}{z} + 1}}\, dz & \zeta < 0 \\ \int_{-c\zeta}^{\frac{1}{\sqrt{2\pi}}} \frac{\sqrt{\frac{2}{\pi}}|c| e^{-\frac{c\zeta}{2z} - \frac{1}{2}}}{z\sqrt{-2\log(z) - \log(2\pi)}\sqrt{\frac{c\zeta}{z} + 1}}\, dz & 0 < \zeta < \frac{-1}{\sqrt{2\pi} c} \end{cases} \tag{5}$$

We can obtain the variance:

$$\mathbb{V}(\zeta) = \frac{1}{\sqrt{3\pi c^2}}$$

IV. The Stochastic Delta Problem

Let us investigate the statistical properties of the hedge to examine what takes place at the limit. Clearly we are dealing with the product of 1) the hedge ratio which follows a uniform distribution on (0,1) and 2) the variations of lognormal S_t at the subsequent time window. The time lag causes independence between the two r.v.s. Both have finite variance, so the sum of the product converges to a Gaussian distribution for which the operation of the (weak) law of large numbers is well known. The distribution of $\overline{\Pi}_n$ is degenerate at the limit, a Gaussian with a variance approaching 0 delivering a Dirac mass at 0.

Let $s\sqrt{\Delta t}$ be the standard deviation for $S_{t_0 + i\Delta t} - S_{(t_0 + (i-1)\Delta t}$, which we assume (in the Black-Scholes world) is normally distributed for small increments (without any loss of generality).

Further, at a small Δt the correlation between the terminal payoff $(S - K)^+$ and $\Psi(\Delta t)$ converges to 0, since, by the properties of a Brownian motion the ratio $\frac{S_t - S_{t_0}}{\sum |\Delta S|}$ becomes small.

It is easy to show that the hedge ratio $h = \frac{\partial}{\partial S}\int_{A_K^c}(S_t - K)\, \mathrm{d}\mu_1$ corresponds to a cumulative probability distribution, which is uniformely distributed on [0,1].

So with:

$$h \sim U(0,1)$$

and

$$\Delta S \sim N(0, s\sqrt{\Delta t})$$

$$\Psi(\Delta t) = h \times \Delta s \sim \frac{\Gamma\left(0, \frac{\Psi(\Delta t)^2}{2s^2 \Delta \mathrm{t}}\right)}{2\sqrt{2\pi}\sqrt{\Delta t} s} \tag{6}$$

The characteristic function of $\Psi(\Delta t)$, $\mathcal{C}(\omega)$, under n-convolutions and time increment $\Delta t/n$:

$$\mathcal{C}(\omega)^n = 2^{-n} |\omega|^{-n} \left(\frac{\operatorname{erf}\left(\frac{s|\omega|\sqrt{\frac{\Delta \mathrm{t}}{n}}}{\sqrt{2}}\right)}{s\sqrt{\frac{\Delta \mathrm{t}}{n}}} \right)^n$$

from which we derive a mean of 0 and a variance:

$$V[\Psi(\Delta t)] = \frac{1}{3} \Delta \mathrm{t} (2\pi)^{-\frac{n}{2}} s^2$$

which in the Black-Scholes world would still converge to 0 (though less rapidly), except for the problem of microstructure, as we see next.

A. Lower bound on Dynamic hedging

The two previous problems set a case for difficulty in convergence. Let us see how in a true dynamic environmnent the limit of a Dirac mass at 0 is never reached.

Let us consider specific periods $t_0, t_1, ...t_n$ over a given increment Δt. The problem, as we see in Figure 1, is that the operator gets his information at period t_1 about the corresponding value of the underlying asset S_{t_1}, then makes an adjustment decision based on the hedge ratio at t_1, since that was the last price he was aware of. The Black Scholes adjustment (Merton, 1973) assumes that the operator is now hedged as of t_2, with the difference in hedging errors remaining of order Δt^2, as we saw, converging to ΔS^2, hence the variance of the asset (in expectation), as part of the properties of stochastic integration.

But Black-Scholes misses another fundamental lag. The operator cannot instantaneously activate an order to buy at t_1, the period in which he gets the information; the order needs to be placed at a subsequent time period $> t_1$, say t_2, with the added complication that the asset price for the revision is obtained as of t_3, two periods later —at the hedge ratio as of t_1, exposed to the variation $\frac{\partial C}{\partial S_t} |S_{t_2}| S_{t_4} - S_{t_2}|$, an incompressible tracking error. The discrepancy is not trivial; it is similar to the difference between the Stratonovich and Ito integrals. The additional incompressible variance prevents the option package from ever having, at the limit of Δt, a purely deterministic payoff.

V. Some comments